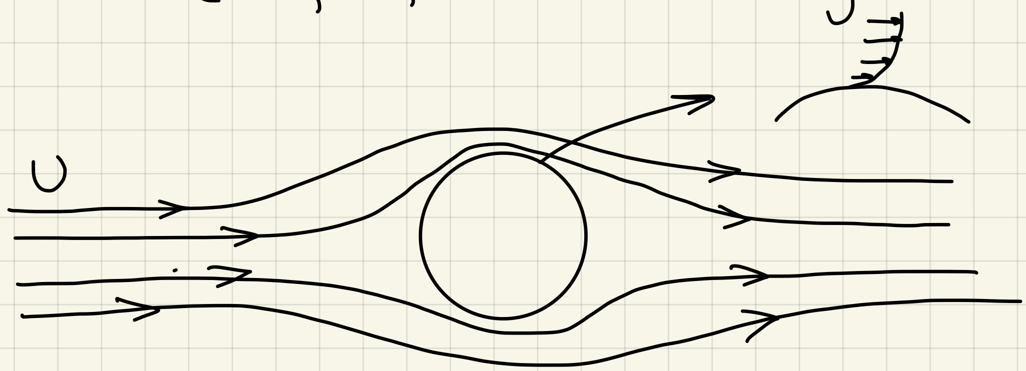
II. Elementary viscous flow

2.2 The problem of 2-D steady viscous flow past a circular cylinder of radius a involves finding a velocity field $\bar{u} = [u(x,y), v(x,y), 0]^T$ which satisfies

$$(\bar{u}\cdot\bar{\nabla})\bar{u} = -\frac{1}{\rho}\bar{\nabla}p + \nu\bar{\nabla}^2\bar{u},$$
$$\bar{\nabla}\,\bar{u} = 0,$$

together with the boundary conditions

$$\bar{u} = 0 \quad \text{on } x^2 + y^2 = a^2,$$
$$\bar{u} \to (U, 0, 0) \quad \text{as } x^2 + y^2 \to \infty.$$



Rewrite this problem in dimensionless form by using the dimensionless variables

$$\bar{x}' = \bar{x}/a, \quad \bar{u}' = \bar{u}/U, \quad p' = p/\rho U^2$$

Without attempting to solve the problem, show that the streamline pattern can depend on ν, a and U only in the combination, so that the flows at equal Reynolds numbers are geometrically similar.

$$(\bar{u}\cdot\bar{\nabla})\bar{u} = -\frac{1}{\rho}\bar{\nabla}p + \nu\bar{\nabla}^2\bar{u} \qquad \frac{\nu}{Ua}$$

$$\Rightarrow \frac{U^{\cancel{2}}}{\cancel{a}}(\bar{u}'\cdot\bar{\nabla}')\bar{u}' = -\frac{U^{\cancel{2}}}{\cancel{a}}\bar{\nabla}'p' + \frac{\nu U}{a^2}\bar{\nabla}'^2\bar{u}'$$

$$\frac{U}{\cancel{a}}\bar{\nabla}'\cdot\bar{u}' = 0$$

$$1/Re$$

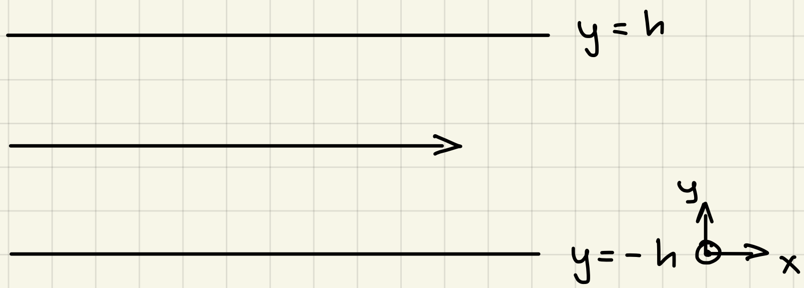
$$\begin{cases} (\bar{u}'\cdot\bar{\nabla}')\bar{u}' = -\bar{\nabla}'p' + \dfrac{\nu}{Ua}\bar{\nabla}'^2\bar{u}' \\ \bar{\nabla}'\cdot\bar{u}' = 0 \end{cases}$$

$$Re = \frac{UL}{\nu}$$

$$\left.\begin{array}{l} L \gg 1 \\ \nu \ll 1 \end{array}\right\} \Rightarrow Re_{exp} = Re_{real}$$

2.3. (i) Viscous fluid flows between two stationary rigid boundaries $y = \pm h$ under a constant pressure gradient $P = -dp/dx$. Show that

$$u = \frac{P}{2\mu}(h^2 - y^2), \quad v = w = 0.$$



$$\bar{u} = (u(y), 0, 0)$$

$$\underbrace{(\bar{u}\cdot\bar{\nabla})u}_{=0} = \underbrace{\frac{P}{\rho}}_{\text{const}} + \nu \frac{d^2u}{dy^2}$$

$$\nu = \mu/\rho$$

$$u'' = -\frac{P}{\rho\nu} = -\frac{P}{\mu}$$

$$u(y) = -\frac{P}{2\mu}y^2 + C_1 y + C_2$$

$$u(-h) = u(h) = 0$$

$$u(h) = -\frac{P}{2\mu}h^2 + C_1 h + C_2 = 0$$

$$u(-h) = -\frac{P}{2\mu}h^2 - C_1 h + C_2 = 0$$

1) $u(h) + u(-h) = -\dfrac{P}{\mu}h^2 + 2C_2 = 0$

$$\Rightarrow C_2 = \frac{P}{2\mu}h^2$$

2) $u(h) - u(-h) = 2C_1 h = 0 \Rightarrow C_1 = 0$

$$u(y) = -\frac{P}{2\mu}y^2 + \frac{P}{2\mu}h^2 = \frac{P}{2\mu}(h^2 - y^2)$$

The plane Poiseuille flow.

(II) Viscous fluid flows down a pipe of circular cross-section $r=a$ under a constant pressure gradient $P=-dp/dz$. Show that

$$u_z = \frac{P}{4\mu}(a^2 - r^2)$$

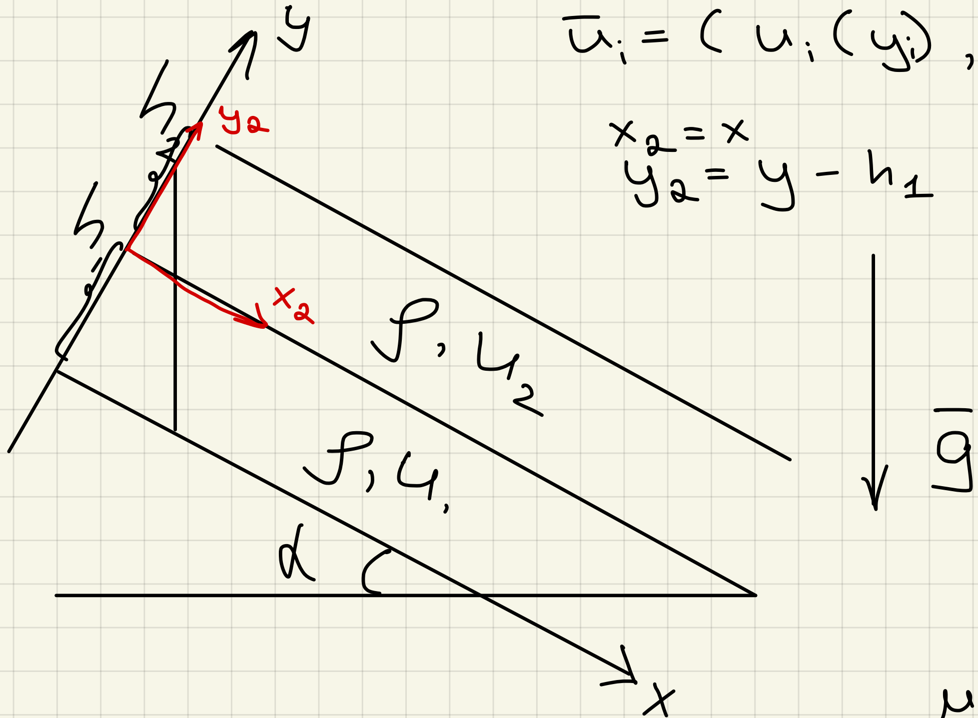
Equivalent solution but in cylindrical coordinate system.

$$\bar{u} = (0, 0, u_z(y))$$

pipe Poiseuille flow.

2.4 Two incompressible viscous fluids of the same density ρ flow, one on top of the other, down an inclined plane making an angle α with the horizontal. Their viscosities are μ_1, and μ_2, the lower fluid is of depth h_1, and the upper fluid is of depth h_2. Show that

$$u_1(y) = \left[(h_1+h_2)y - \frac{1}{2}y^2\right]\frac{g\sin\alpha}{\nu_1}$$



$$\bar{u}_i = (u_i(y_i), 0, 0), \quad i = 1, 2$$

$$x_2 = x$$
$$y_2 = y - h_1$$

$$\bar{g} = (g\sin\alpha, -g\cos\alpha, 0)$$

$$\left\{\begin{array}{l} \cancel{(\bar{u}_i \bar{\nabla}_i)u_i} = -\dfrac{1}{\rho}\dfrac{\partial p_i}{\partial x} + \overset{\mu_i/\rho}{\nu_i}\, u_i'' + g\sin\alpha \\ \underbrace{\dfrac{1}{\rho}\dfrac{\partial p_i}{\partial y_i} = -g\cos\alpha} \end{array}\right.$$

$$p_i = -\rho g y\cos\alpha + f_i(x)$$

$$-\frac{1}{\rho}f_i' + \nu_i u_i'' + g\sin\alpha = 0$$

$$\frac{1}{\rho}\frac{df}{dx} = \nu_i\frac{d^2u_i}{dy_i^2} + g\sin\alpha \neq \mathrm{func}(x)$$

$$\frac{df}{dx} = \mathrm{const}$$

$$\nu_i u_i'' = \underbrace{-g\sin\alpha - K}_{\mathrm{const}} = -H$$

$$u_i'' = -\frac{H}{\nu_i} = -H_i$$

$$u_i = -\frac{H_i}{2}y_i^2 + C_{1,i}\,y_i + C_{2,i}$$

(I) "No-slip" at rigid bed:

$$u_1(0) = 0$$

$$u_1(0) = C_{2,1} = 0$$

(II) "Free-surf." cond.:

$$u_2'(y_2 = h_2) = 0$$

$$u_2'(h_2) = -H_2 h_2 + C_{1,2} = 0$$

$$C_{1,2} = H_2 h_2$$

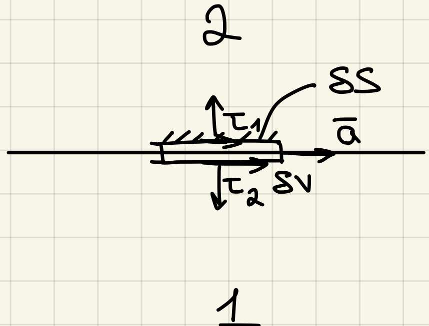
(III) "No-slip" between 2 fluids:

$$u_1 = u_2 \quad \text{on } y = h_1 \text{ or } y_2 = 0$$

$$-\frac{H_1}{2}h_1^2 + C_{1,1}h_1 = C_{2,2}$$

(IV) the shear stress exerted on the fluid 1 by the fluid 2 = the sh. str. ex. on the fluid 2 by the fluid 1.

$$\nu_1 u_1' = \nu_2 u_2' \quad \text{on } y = h_1 \text{ or } y_2 = 0$$



$$\tau_1\,\delta\bar S - \tau_2\,\delta\bar S = \rho\,\delta V\,\bar a$$

$$\bar a = \frac{\tau_1\,\delta\bar S - \tau_2\,\delta\bar S}{\rho\,\underbrace{\delta V}_{\underbrace{\delta e\,\delta S}_{\to\, 0}}}$$

$$\Rightarrow \bar a \to \infty$$

$$\Rightarrow \tau_1 = \tau_2$$

$$-\nu_1 H_1 h_1 + \nu_1 C_{1,1} = \nu_2 C_{1,2} = \nu_2 H_2 h_2$$

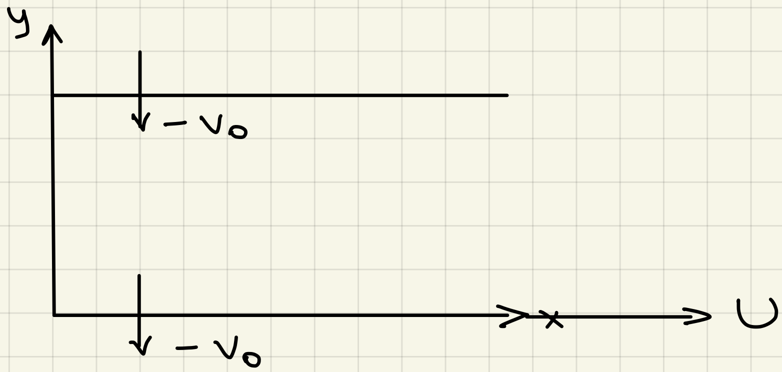
$$C_{1,1} = \frac{h_1\nu_1 H_1 + \nu_2 H_2 h_2}{\nu_1} = \frac{H(h_1+h_2)}{\nu_1}$$

$$u_1(y) = -\frac{H}{2\nu_1}\, y^2 + \frac{H(h_1 + h_2)}{\nu_1}\, y$$

$$= \frac{(g \sin\alpha + \cancel{K})}{\nu_1}\left[(h_1 + h_2)\, y - \frac{1}{2} y^2 \right]$$

2.6 Viscous fluid flows between two rigid boundaries $y=0, y=h$, the lower boundary moving in the x-direction with constant speed U, the upper boundary being at rest. The boundaries are porous, and the vertical velocity v is $-V_0$ at each one, V_0, being a given constant (so that there is an imposed flow across the system). Show that the resulting flow is

$$u = U\left(\frac{e^{-V_0 y/\nu} - e^{-V_0 h/\nu}}{1 - e^{-V_0 h/\nu}}\right), \quad v = -V_0$$



$$\bar{u} = (u(y), -V_0, 0)$$

$$(\bar{u}\cdot\bar{\nabla})u = \nu u''$$

$$u(0) = U, \quad u(h) = 0$$

$$-V_0 u' = \nu u''$$

$$\frac{u''}{u'} = -\frac{V_0}{\nu} = (\ln u')'$$

$$\ln u' = -\frac{V_0}{\nu} y + \tilde{C}_1$$

$$u' = C_1 e^{-V_0 y/\nu}$$

$$u(y) = -\frac{\nu}{V_0} C_1 e^{-V_0 y/\nu} + C_2$$

$$u(0) = -\frac{\nu}{V_0} C_1 + C_2 = U$$

$$u(h) = -\frac{\nu}{V_0} C_1 e^{-V_0 h/\nu} + C_2 = 0$$

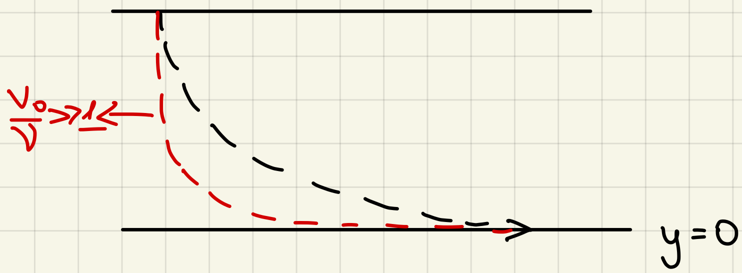
$$C_2 = \frac{\nu}{V_0} C_1 e^{-V_0 h/\nu}$$

$$-\frac{\nu}{V_0} C_1 + \frac{\nu}{V_0} C_1 e^{-V_0 h/\nu} = U$$

$$C_1 = \frac{U v_0}{\nu} \frac{1}{e^{-v_0 h/\nu} - 1}$$

$$u(y) = -\frac{\cancel{\nu}}{\cancel{v_0}} \cdot \frac{U \cancel{v_0}}{\cancel{\nu}} \frac{1}{e^{-v_0 h/\nu} - 1} e^{-v_0 y/\nu} + \frac{\cancel{\nu}}{\cancel{v_0}} \cdot \frac{U \cancel{v_0}}{\cancel{\nu}} \frac{e^{-v_0 h/\nu}}{e^{-v_0 h/\nu} - 1}$$

$$u(y) = \frac{e^{-v_0 y/\nu} - e^{-v_0 h/\nu}}{1 - e^{-v_0 h/\nu}} U \sim e^{-v_0 y/\nu}$$



rate of decay $\sim \frac{v_0}{\nu}$

$v_0/\nu >> 1$

1) $v_0 >> 1 \rightarrow$ fast movement throug the walls

2) $\nu << 1$

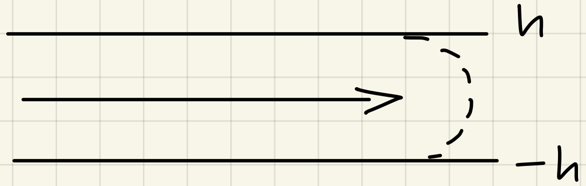
$$Re = \frac{v_0 h}{\nu}$$

$$u(y) = \frac{e^{-Re \frac{y}{h}} - e^{-Re}}{1 - e^{-Re}}, \quad \frac{y}{h} \sim y'$$

2.5 Viscous fluid is at rest in a two-dimensional channel between two stationary rigid wals $y=\pm h$. For $t>0$, a constant pressure gradient $P=-\frac{dp}{dx}$ is imposed. Show that $u(y,t)$ satisfies

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}+\frac{P}{\rho},$$

and give suitable initial and boundary conditions. Find $u(y,t)$ in the form of a Fourier series, and show that the flow approximates to steady channel flow when $t \gg h^2/\nu$.



Plane Poiseuille flow:

$$u(y)=\frac{P}{2\mu}\left(h^2-y^2\right)$$

$t=0$	$0<t<t_f$	$t \geq t_f$
$\bar{u}=0$	$\bar{u}(y,t)=u(y,t)\bar{e}_x$	$\bar{u}(y)$

Eq-n of motion:

$$\frac{\partial u}{\partial t}=\frac{P}{\rho}+\nu\frac{\partial^2 u}{\partial y^2} \qquad (*)$$

BC:

$$u(-h,t)=u(h,t)=0$$

Init. cond.:

$$u(y,0)=0.$$

$$u(y,t)=\underbrace{u_0(y)}_{\text{Pois. flow}}+u_1(y,t)$$

$$\frac{\partial u_1}{\partial t}=\underbrace{\frac{P}{\rho}+\nu\frac{d^2u_0}{dy^2}}_{=0}+\nu\frac{\partial^2 u_1}{\partial y^2}$$

$$\frac{\partial u_1}{\partial t}=\nu\frac{\partial^2 u_1}{\partial y^2} \qquad (**)$$

$$u_1(-h,t) = u_1(h,t) = 0$$

$$u_1(y,0) = u(y,0) - u_0(y) = 0 - \frac{P}{2\mu}(h^2-y^2)$$

$$= -\frac{P}{2\mu}(h^2-y^2)$$

$$u_1(y,t) = Y(y)\,T(t)$$

$$YT' = \nu Y''T$$

$$\Rightarrow \frac{Y''}{Y} = \frac{T'}{T}\frac{1}{\nu} = C$$

$$\begin{cases} Y'' - CY = 0 \\ T' - \nu C T = 0 \end{cases}$$

1) $Y'' - CY = 0$

- $C > 0$, $C = \lambda^2$

$$Y = K_1 e^{\lambda y} + K_2 e^{-\lambda y}$$

$$Y(-h) = K_1 e^{-\lambda h} + K_2 e^{\lambda h} = 0 \rightarrow \text{div. by } e^{-\lambda h}$$

$$Y(h) = K_1 e^{\lambda h} + K_2 e^{-\lambda h} = 0$$

$$\Rightarrow K_1 + K_2 e^{2\lambda h} = 0$$

$$-K_2 e^{2\lambda h} e^{\lambda h} + K_2 e^{-\lambda h} = -K_2 e^{-\lambda h}(e^{4\lambda h} - 1) = 0$$

$$\Rightarrow K_2 = K_1 = 0 \rightarrow \text{only triv. solut.}$$

- $C = 0 \Rightarrow Y'' = 0$

$$\Rightarrow Y = K_1 + K_2 y$$

$$Y(-h) = K_1 - K_2 h = 0$$
$$Y(h) = K_1 + K_2 h = 0$$

$$K_1 = K_2 = 0 \rightarrow \text{only triv. sol.}$$

- $C < 0$, $C = -\lambda^2 < 0$

$$Y = K_1 \cos\lambda y + K_2 \sin\lambda y$$

$$y(-h) = K_1 \cos\lambda h - K_2 \sin\lambda h = 0$$
$$y(h) = K_1 \cos\lambda h + K_2 \sin\lambda h = 0$$

$$K_1 \cos\lambda h = 0$$
$$K_2 \sin\lambda h = 0$$

$\underline{\cos\lambda h = 0}$, $K_2 = 0$

$\hookrightarrow \lambda h = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \dots$

$$y_n = K_{1,n} \cos\left[(n + \tfrac{1}{2})\pi \frac{y}{h}\right]$$

2) $T' - \nu C_n T = 0$, $c_n < 0$

$$T_n = M_n \exp\left[-(n + \tfrac{1}{2})^2 \frac{\pi^2}{h^2} \nu t\right]$$

$$u_{1,n}(y,t) = \underbrace{A_n}_{= K_{1,n} M_n} \cos\left[(n + \tfrac{1}{2})\pi \frac{y}{h}\right] \exp\left[-(n + \tfrac{1}{2})^2 \frac{\pi^2}{h^2} \nu t\right]$$

$$u_1(y,t) = \sum_{n=0}^{\infty} A_n \cos\left[(n + \tfrac{1}{2})\pi \frac{y}{h}\right] \exp\left[-(n + \tfrac{1}{2})^2 \frac{\pi^2}{h^2} \nu t\right]$$

1) $u_1(y,0) = \sum_n A_n \cos\left[(n + \frac{1}{2})\pi \frac{y}{h}\right]$

Multiply the LHS and the RHS by $\cos\left[(m + \frac{1}{2})\pi \frac{y}{h}\right]$

$m = n$ or $m \neq n$

2) Integr. $\int_{-h}^{h} dy$

$\Rightarrow$ on the RHS the only non-zero term for $n = m$

$$\int_{-h}^{h} \frac{P}{2\mu}(y^2 - h^2) \cos\left[(m + \tfrac{1}{2})\frac{\pi}{h} y\right] dy$$
$$= \int_{-h} A_m \cos^2\left[(m + \tfrac{1}{2})\frac{\pi}{h} y\right] dy$$

2.7 Incompressible fluid occupies the space $0<y<\infty$ above a plane rigid boundary $y=0$ which oscillates to and fro in the x-direction with velocity $U\cos\omega t$ Show that the velocity field $\bar{u}=[u(y,t),0,0]$ satisfies

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$$

(there being no applied pressure gradient), and by seeking a solution of the form

$$u=Re[f(y)e^{i\omega t}],$$

where Re denotes 'real part of', show that

$$u(y,t)=Ue^{-ky}\cos(ky-\omega t),$$

where $k=(\omega/2\nu)^{1/2}$.

$\bar{u}_w = U\cos\omega t\,\bar{e}_x \rightarrow \bar{u}=[u(y,t),0,0]$

BC: $u(0,t)=U\cos\omega t$,

$u(\infty,t)=0$.

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2} \quad (*)$$

$$u=\underbrace{Re}_{\text{real part}}[f(y)e^{i\omega t}] \rightarrow \text{normal mode sol.}$$

$$i\omega f\cancel{e^{i\omega t}}=\nu f''\cancel{e^{i\omega t}}$$

$$\Rightarrow f''=\frac{i\omega}{\nu}f$$

General sol.: $f=Ae^{\sqrt{\frac{i\omega}{\nu}}y}+Be^{-\sqrt{\frac{i\omega}{\nu}}y}$.

$$\sqrt{i} = \pm \frac{1}{\sqrt{2}} (1 + i).$$

$$k = \sqrt{\frac{\omega}{2\nu}}$$

$$\Rightarrow f(y) = Ae^{(1+i)ky} + Be^{-(1+i)ky}$$

$$BC: f(y \to \infty) = Ae^{(1+i)k\infty} = 0 \Rightarrow A = 0$$

$$u(y,t) = Re\left[Be^{-(1+i)ky} e^{i\omega t} \right]$$

$$= Re\left[Be^{-ky} e^{i(\omega t - ky)} \right]$$

$$= Be^{-ky} \cos(\omega t - ky)$$

$$u(0,t) = B \cos \omega t = U \cos \omega t$$

$$\Rightarrow B = U$$

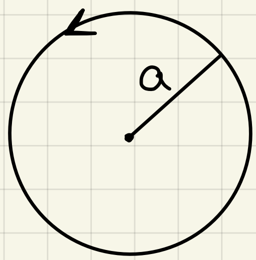
$$u(y,t) = Ue^{-ky} \cos(\omega t - ky)$$

2.8 A circular cylinder of radius a rotates with constant angular velocity Ω in a viscous fluid. Show that the line vortex flow

$$\bar{u} = \frac{\Omega a^2}{r} \bar{e}_\theta, \text{ for } r \geq a$$

is an exact solution of the equations and boundary conditions.

Describe roughly how the vorticity changes with time when the cylinder is suddenly started into rotation from a state of rest. Likewise, discuss the case in which an outer cylinder $r = b$ is simultaneously given an angular velocity $\Omega a^2/b^2$.



surrounded by fluid.

$$\bar{u}(r,t) = u(r,t)\bar{e}_\theta$$

BC: $u(\infty, t) = 0$, $u(a,t) = \Omega a$

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right)$$

$$\bar{u}_v = \frac{\Omega a^2}{r}\bar{e}_\theta \text{ for } r \geq a.$$

$$u_v(a) = \Omega a$$

$$\frac{\partial u_v}{\partial t} = 0$$

$$+\frac{2\Omega a^2}{r^3} - \frac{\Omega a^2}{r^3} - \frac{\Omega a^2}{r^3} = 0$$

$$\bar{\omega} = \bar{\nabla} \times \bar{u} = \bar{\omega}(r,t)$$

$$\Rightarrow \frac{\partial \bar{\omega}}{\partial t} = \nu \bar{\nabla}^2 \bar{\omega}$$

$$\bar{\omega}_v = \bar{e}_z\left(\frac{\partial u_v}{\partial r} + \frac{u_v}{r}\right) = 0$$